Experimental Results

The performance of the matching device depends upon the termination of the filled guide. Two termination methods were used in obtaining experimental data. With the ruby- or alumina-filled guides, we used a termination made from a polyiron-filled waveguide, which absorbs the power propagating down the waveguide. Fig. 4 shows the transition to an alumina-filled waveguide and the polyiron termination; the corresponding VSWR is shown in Fig. 5. The rutile-filled waveguide was more difficult to terminate with an absorbing material. Polyiron has an insufficiently high dielectric constant to be well matched to rutile. Titania with a 20 per cent doping of silicon carbide offers promise as an absorbing material for titania- and rutile-filled waveguides and is presently being tested.

For the test data for titania reported here, a second method of terminating the filled waveguide was used. In this method, two transitions are used—one to match into the filled waveguide, and one to match out to the air-filled waveguide and then to a standard termination (see Fig. 6). The titania-filled waveguide is 0.040 in in the b dimension and 0.090 in in the a dimension. The VSWR plot for this configuration is shown in Fig. 7. Since the measured reflected power is from both transitions, the data in Fig. 7 were those calculated for each transition. Performance data for some typical transitions appear in Table I.

Conclusion

It has been shown that a broad-band RF impedance match from air-filled to dielectric-filled waveguide is possible using an abrupt transition with an appropriate susceptance match. The technique described offers considerable improvement in performance over more conventional dielectric taper transitions, and is easier to fabricate. Immediate application is seen in maser circuits where the waveguide is filled with high dielectric maser material.

Excitation of Plasma Waves in an Unbounded Homogeneous Plasma by a Line Source*

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Summary—**The radiation characteristics of a line source of magnetic current embedded in a homogeneous electron plasma of infinite extent are investigated for the case in which a uniform magnetic field is impressed externally throughout the medium in the direction of the source. The single-fluid theory of magnetohydrodynamics is employed. A very simple model is assumed for the plasma. Under this assumption, it is found that there are two modes of propagation of waves of small amplitude. By examining the behavior of these modes in the limiting cases of vanishing external magnetic field or infinite source frequency, they are identifiable as the modified forms of the usual plasma and optical modes which exist in an isotropic electron plasma. The dispersion relations for these two modes are discussed. The power radiated in each of the two modes is also evaluated. It is found that the power radiated in the optical mode is always lower than that due to the line source in free space, whereas the power radiated in the plasma mode is higher than that value for certain ranges of the source frequency.**

Introduction

THE STUDY OF the radiation characteristics of localized electromagnetic sources in an unbounded ionized gaseous medium, known generally as plasma, has application to the problem of radio communication with missiles at the time of their re-entry into the earth's atmosphere and with space vehicles passing through the ionosphere and other ionized regions in interplanetary space. In recent years, this subject has received considerable attention in literature. Previous investigations of this subject may be conveniently grouped into three categories.

In the first category, the plasma is assumed to be incompressible so that the presence of the longitudinal plasma waves is ruled out. Under this assumption, the plasma reduces to a dielectric medium characterized by a tensor dielectric constant. In the absence of an external static magnetic field, the tensor dielectric constant becomes a scalar. The characteristics of plane wave propagation in such an anisotropic dielectric medium have been studied, but without taking into account the sources which excite these waves. Also, the radiation characteristics of sources in a plasma idealized by an anisotropic dielectric medium were investigated. For example, Arbel[1] has treated the problem of radiation from a point source in an incompressible homogeneous plasma medium of infinite extent.

* Received August 27, 1962. The research reported in this paper was supported by the National Science Foundation Grant NSF G-9721.

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[1] E. Arbel, "Radiation from a Point Source in an Anisotropic Medium," Polytechnic Inst. of Brooklyn, N. Y., Res. Rept. No. PIBMRI-861-60; November, 1960.

In the second category of papers published on this subject, the compressibility of the medium is taken into account and consequently the presence of the longitudinal plasma waves as well as that of the transverse electromagnetic waves is included. The properties of plane waves propagating in an unbounded plasma medium without an external magnetic field have been studied,[2] but ignoring the presence of any sources. The longitudinal plasma waves and the transverse electromagnetic waves interact with each other on the application of an external magnetic field. The plane wave characteristics of these modified plasma and electromagnetic waves have also been examined.[3]

The third category of papers takes into account both the compressibility of the medium and the presence of sources of excitation. Hessel and Shmoys[4] have treated the problem of radiation from a point source of electric current in a homogeneous isotropic plasma. In the case of an isotropic plasma, that is, in the absence of an external magnetic field, with the introduction of the so-called "modified" electric field, the "modified" electromagnetic field and the pressure are found to satisfy two separate wave equations which are coupled only through the source term. On the application of an external magnetic field, the electromagnetic field and the pressure cease to satisfy separate and simple wave equations. Consequently, the problem becomes more difficult and has not been treated in the literature. In this paper a simple case of such a problem, namely, the radiation characteristics of a line source of magnetic current in a compressible plasma medium of infinite extent under an applied static magnetic field in the direction of the source is investigated.

Formulation of the Problem

Consider a homogeneous electron plasma of infinite extent. It is desired to examine the radiation characteristics of a line source of magnetic current in this medium. It is convenient to introduce a right-handed rectangular coordinate system x, y, and z. In this system, the line source is taken to be along the y axis, and hence, it is given by

$$J_m = \hat{y} J_0 \delta(x) \delta(z). \tag{1}$$

Only the steady-state problem is considered, and the current source is assumed to have a harmonic time dependence of the form $e^{-i\omega t}$. The frequency of the source is assumed to be sufficiently high so that the ions may be considered stationary. Hence, in this treatment the plasma medium reduces to that of a gas of electrons whose motion introduces coupling with the electromagnetic field. The amplitude of all the excited waves is considered to be small, thus justifying the use of a linearized plasma theory.[5] The use of the linearized theory implies that all the field components will have the same harmonic time dependence as that of the source, namely, $e^{-i\omega t}$ which may, therefore, be conveniently suppressed. The collisions between electrons and other particles are neglected. It is further assumed that the drift velocity of electrons is zero so that the plasma as whole may be considered as stationary. A uniform magnetic field B_0 is assumed to be impressed externally throughout the plasma region in the y direction which is parallel to that of the source.

Let N_0 be the average number density, p the pressure deviation from the mean, and v the velocity of the electrons. Let $\boldsymbol{E}$ and $\boldsymbol{H}$ be the alternating electric and magnetic fields. It is to be noted that $\boldsymbol{v}$, $\boldsymbol{E}$ and $\boldsymbol{H}$ are small perturbations. The linearized time-harmonic hydrodynamic equation of motion for the electrons is

$$-i\omega m N_0 \boldsymbol{v} = N_0 e(\boldsymbol{E} + \boldsymbol{v} x \hat{y} B_0) - \nabla p \tag{2}$$

where e is the charge and m is the mass of an electron. The equation of continuity after being linearized and combined with the equation of state is given by

$$a^2 m N_0 \nabla \cdot \boldsymbol{v} = i\omega p \tag{3}$$

where a is the velocity of sound in the electron gas. In addition, the electric and the magnetic fields satisfy the following time-harmonic Maxwell's equations

$$\nabla \times \boldsymbol{E} = i\omega\mu_0 \boldsymbol{H} - \boldsymbol{J}_m \tag{4}$$

$$\nabla \times \boldsymbol{H} = -i\omega\epsilon_0 \boldsymbol{E} + N_0 e \vec{v}, \tag{5}$$

where μ_0 and ϵ_0 are the permeability and the dielectric constant of free space.

The source and the geometry of the problem are independent of the y coordinate and, therefore, all the field quantities are invariant with respect to the y coordinate. Hence, the y component of the particle velocity v_y is zero. On substituting $\partial/\partial y = 0$ in (4) and (5), it is found that the electromagnetic field is separable into E and H modes which are known to be excited, respectively, by line sources of magnetic and electric current. Since only a line source of magnetic current is present, the H mode is not excited, and hence, $E_y = H_x = H_z = 0$. Only a single component of the magnetic field, namely H_y, is present.

On writing (2) in component form and noting that $v_y = E_y = 0$, two simultaneous equations in v_x and v_z are obtained in terms of E_x, E_z, $\partial p/\partial x$ and $\partial p/\partial z$. The result of the solution of these equations for v_x and v_z is

$$v_x = \frac{ie}{\omega m \alpha} E_x + \frac{e^2 B_0}{\omega^2 m^2 \alpha} E_z - \frac{i}{\omega m N_0 \alpha} \frac{\partial p}{\partial x} - \frac{e B_0}{\omega^2 m^2 N_0 \alpha} \frac{\partial p}{\partial z} \tag{6}$$

$$v_z = -\frac{e^2 B_0}{\omega^2 m^2 \alpha} E_x + \frac{ie}{\omega m \alpha} E_z + \frac{e B_0}{\omega^2 m^2 N_0 \alpha} \frac{\partial p}{\partial x} - \frac{i}{\omega m N_0 \alpha} \frac{\partial p}{\partial z}, \tag{7}$$

[2] S. I. Pai, "Wave motions of small amplitude in a fully-ionized plasma without external magnetic field," *Revs. Mod. Phys.*, vol. 32, pp. 882–887; October, 1960.

[3] S. I. Pai, "Wave motions of small amplitude in a fully-ionized plasma under applied magnetic field," *Phys. of Fluids*, vol. 5, pp. 234–240; February, 1962.

[4] A. Hessel and J. Shmoys, "Excitation of Plasma Waves in a Homogeneous Isotropic Plasma by a Dipole," Polytechnic Inst. of Brooklyn, N. Y., Memo. 63, PIBMRI-921-61; July, 1961.

[5] L. Oster, "Linearized theory of plasma oscillations," *Revs. Mod. Phys.*, vol. 32, pp. 141–168; January, 1960.

where

$$\alpha = 1 - \left(\frac{\omega_c}{\omega}\right)^2$$

and the electron gyromagnetic frequency ω_c is given by

$$\omega_c = -\frac{eB_0}{m}. \tag{9}$$

If Maxwell's second equation (5) is written in component form and the expressions (6) and (7) are substituted, respectively, for v_x and v_z, a pair of simultaneous equations for E_x and E_z is obtained in terms of H_y and p. When these equations are solved for E_x and E_z, the following result is obtained:

$$E_x = \frac{\epsilon_1}{i\omega\epsilon_0\epsilon}\frac{\partial H_y}{\partial z} - \frac{\epsilon_2}{\omega\epsilon_0\epsilon}\frac{\partial H_y}{\partial x} - \frac{(\epsilon_1 - \epsilon)}{N_0 e\epsilon}\frac{\partial p}{\partial x} - \frac{i\epsilon_2}{N_0 e\epsilon}\frac{\partial p}{\partial z} \tag{10}$$

$$E_z = -\frac{\epsilon_1}{i\omega\epsilon_0\epsilon}\frac{\partial H_y}{\partial x} - \frac{\epsilon_2}{\omega\epsilon_0\epsilon}\frac{\partial H_y}{\partial z} + \frac{i\epsilon_2}{N_0 e\epsilon}\frac{\partial p}{\partial x} - \frac{(\epsilon_1 - \epsilon)}{N_0 e\epsilon}\frac{\partial p}{\partial z} \tag{11}$$

where

$$\epsilon = \epsilon_1^2 - \epsilon_2^2 \tag{12}$$

$$\epsilon_1 = 1 - \frac{\omega_p^2}{\omega^2\alpha} \tag{13}$$

and

$$\epsilon_2 = \frac{\omega_c\omega_p^2}{\omega^3\alpha} \tag{14}$$

and ω_p in (13) and (14) is the plasma frequency and is given by

$$\omega_p^2 = \frac{N_0 e^2}{\epsilon_0 m}. \tag{15}$$

If in (6) and (7), E_x and E_z are replaced by the expressions given in (10) and (11), the following expressions are obtained for v_x and v_z in terms of H_y and p. After some simplification, they are

$$v_x = -\frac{ie\omega_c}{\omega^3 m\epsilon_0\epsilon\alpha}\left\{\frac{\partial H_y}{\partial x} + \frac{i\omega\epsilon_0}{N_0 e}\frac{\partial p}{\partial z}\right\} + \frac{e\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]}{\omega^2 m\epsilon_0\epsilon a}\left\{\frac{\partial H_y}{\partial z} - \frac{i\omega\epsilon_0}{N_0 e}\frac{\partial p}{\partial x}\right\}. \tag{16}$$

$$v_z = -\frac{e\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]}{\omega^2 m\epsilon_0\epsilon\alpha}\left\{\frac{\partial H_y}{\partial x} + \frac{i\omega\epsilon_0}{N_0 e}\frac{\partial p}{\partial z}\right\} - \frac{ie\omega_c}{\omega^3 m\epsilon_0\epsilon\alpha}\left\{\frac{\partial H_y}{\partial z} - \frac{i\omega\epsilon_0}{N_0 e}\frac{\partial p}{\partial x}\right\}. \tag{17}$$

The substitution of (16) and (17) in (3) gives the following result:

$$\frac{\left(\frac{\omega_p}{\omega}\right)^2}{\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]} B_0\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)H_y(x, z) - \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2\alpha\epsilon}{a^2\left(1 - \frac{\omega_p^2}{\omega^2}\right)}\right]p(x, z) = 0. \tag{18}$$

In a similar way, and after some manipulation, the use of (10) and (11) in (4) leads to the following equation:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2\mu_0\epsilon_0\epsilon}{\epsilon_1}\right]H_y(x, z) + \frac{\omega\epsilon_0\epsilon_2}{N_0 e\epsilon_1}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]p(x, z) = -\frac{i\omega\epsilon_0\epsilon}{\epsilon_1}J_0\delta(x)\delta(z). \tag{19}$$

Eqs. (18) and (19) are the coupled differential equations for $H_y(x, z)$ and $p(x, z)$. Once these quantities are determined, (10, (11), (16), and (17) can be used to obtain E_x, E_z, v_x, and v_z. Notice that the so-called electromagnetic mode given by $H_y(x, z)$ and the plasma mode given by $p(x, z)$ are coupled.

Before proceeding to solve the coupled-wave equations (18) and (19), it is instructive to examine their behavior in the following three limiting cases: 1) when the plasma is incompressible, so that $p=0$; 2) when the applied magnetic field B_0 becomes zero; and 3) when the source frequency becomes extremely large. In the first case, (19) reduces to

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2\mu_0\epsilon_0\epsilon}{\epsilon_1}\right]H_y(x, z) = -\frac{i\omega\epsilon_0\epsilon}{\epsilon_1}J_0\delta(x)\delta(z). \tag{20}$$

This wave equation (20) has been obtained and used previously.[6] In the second case, (18) and (19) can be simplified to yield the following equations:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{a^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\right]p(x, z) = 0 \tag{21a}$$

[6] S. R. Seshadri, "Excitation of Surface Waves on a Perfectly-Conducting Screen Covered with Anisotropic Plasma," Cruft Lab., Harvard Univ., Cambridge, Mass., Tech. Rept. 366; May, 1962.

and

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p{}^2}{\omega^2}\right)\right] H_y(x, z) = -i\omega\epsilon_0\left(1 - \frac{\omega_p{}^2}{\omega^2}\right) J_0\delta(x)\delta(z) \quad (21b)$$

where $c = 1/\sqrt{\mu_0\epsilon_0}$ is the velocity of electromagnetic waves in free space. Hence, in the absence of an external magnetic field, the plasma mode given by (21a) and the electromagnetic mode given by (21b) are uncoupled. Moreover, any coupling between these modes can take place only at a boundary, as can be seen from the expressions for E_x, E_z, v_x, and v_z given in (10, (11), (16), and (17), respectively. Variants of the specialized and simple equations (21a) and (21b) have been used previously by Hessel, Marcuvitz, and Shmoys[7] who have studied the coupling between the plasma and the electromagnetic modes at a vacuum-plasma interface. In the third case of an infinitely large source frequency, the coupled equations again become uncoupled and the following two separate equations are obtained:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{a^2}\right] p(x, z) = 0, \quad (22a)$$

and

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right] H_y(x, z) = -i\omega\epsilon_0 J_0\delta(x)\delta(z). \quad (22b)$$

As in the previous case, the two modes are coupled only at the boundary. The behavior of the modes in the limiting cases of vanishing external magnetic field and infinite source frequency will be used later and identified as modified forms of the separate plasma and electromagnetic modes of the isotropic plasma.

Fourier-Transform Solution of (18) and (19)

It is proposed to solve (18) and (19) for $H_y(x, z)$ and $p(x, z)$ by the method of Fourier transform. For this purpose, let the following Fourier transforms be defined:

$$\overline{\overline{H}}_y(\zeta, \eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_y(x, z) e^{-i(\zeta x + \eta z)} dx dz \quad (23)$$

$$H_y(x, z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\overline{H}}_y(\zeta, \eta) e^{i(\zeta x + \eta z)} d\zeta d\eta \quad (24)$$

$$\overline{\overline{p}}(\zeta, \eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, z) e^{-i(\zeta x + \eta z)} dx dz \quad (25)$$

and

$$p(x, z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\overline{p}}(\zeta, \eta) e^{i(\zeta x + \eta z)} d\zeta d\eta. \quad (26)$$

[7] A. Hessel, N. Marcuvitz, and J. Shmoys, "Scattering and guided waves at an interface between air and a compressible plasma," IRE Trans. on Antennas and Propagation, vol. AP-10, pp. 48–54; January, 1962.

On applying Fourier transforms to (18) and (19) with respect to x and z, it follows that

$$\frac{\left(\frac{\omega_p}{\omega}\right)^2}{\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]} B_0(\zeta^2 + \eta^2)\overline{\overline{H}}_y(\zeta, \eta) + [-\zeta^2 - \eta^2 + k_a{}^2]\overline{\overline{p}}(\zeta, \eta) = 0 \quad (27)$$

and

$$[-\zeta^2 - \eta^2 + k_e{}^2]\overline{\overline{H}}_y(\zeta, \eta) + \frac{\omega\epsilon_0\epsilon_2}{N_0 e\epsilon_1}[-\zeta^2 - \eta^2]\overline{\overline{p}}(\zeta, \eta) = -\frac{i\omega\epsilon_0\epsilon}{\epsilon_1} J_0, \quad (28)$$

where the following short-hand notation has been employed:

$$k_a{}^2 = \frac{\omega^4\alpha\epsilon}{a^2(\omega^2 - \omega_p{}^2)} \quad (29)$$

and

$$k_e{}^2 = \frac{\omega^2}{c^2}\frac{\epsilon}{\epsilon_1}. \quad (30)$$

The solution of (27) and (28) for $\overline{\overline{p}}(\zeta, \eta)$ and $\overline{\overline{H}}_y(\zeta, \eta)$ gives

$$\overline{\overline{p}}(\zeta, \eta) = \frac{\left(\frac{\omega_p}{\omega}\right)^2}{\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]} B_0(\zeta^2 + \eta^2)\frac{i\omega\epsilon_0\epsilon}{\epsilon_1}\frac{J_0}{\Delta} \quad (31)$$

and

$$\overline{\overline{H}}_y(\zeta, \eta) = \frac{i\omega\epsilon_0\epsilon}{\epsilon_1}(\zeta^2 + \eta^2 - k_a{}^2)\frac{J_0}{\Delta}, \quad (32)$$

where

$$\Delta = [\zeta^2 + \eta^2 - k_e{}^2][\zeta^2 + \eta^2 - k_a{}^2] - \beta(\zeta^2 + \eta^2)^2 \quad (33)$$

and

$$\beta = \frac{1}{1 - \frac{\omega_p{}^2}{\omega^2}}\frac{\omega_c}{\omega}\frac{\epsilon_2}{\epsilon_1}. \quad (34)$$

In obtaining (33) and (34) from (27) and (28), some simplification has been effected by the use of (9) and (15).

The substitution of (32) and (31), respectively in (24) and (26), gives

$$H_y(x, z) = \frac{1}{(2\pi)^2}\frac{i\omega\epsilon_0\epsilon}{\epsilon_1} J_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\eta^2 + \zeta^2 - k_a{}^2)}{\Delta} \cdot e^{i(\zeta x + \eta z)} d\zeta d\eta \quad (35)$$

and

$$p(x, z) = \frac{1}{(2\pi)^2} \frac{\left(\frac{\omega_p}{\omega}\right)^2}{\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]} B_0 \frac{i\omega\epsilon_0\epsilon}{\epsilon_1} J_0$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\eta^2 + \zeta^2)}{\Delta} e^{i(\zeta x + \eta z)} d\zeta d\eta. \quad (36)$$

For the evaluation of the double integrals, the integration with respect to η is performed first. In order to carry out this integration, the singularities of Δ considered as a function of η are needed. It is possible to put Δ in the following form:

$$\Delta = (1 - \beta)(\eta^2 + \zeta^2 - k_{mp}^2)(\eta^2 + \zeta^2 - k_{m0}^2) \quad (37)$$

where

$$k_{m0}^2 = U - \sqrt{U^2 - W} \quad (38)$$

$$k_{mp}^2 = U + \sqrt{U^2 - W} \quad (39)$$

$$U = \frac{k_a^2 + k_e^2}{2(1 - \beta)} \quad (40)$$

and

$$W = \frac{k_a^2 k_e^2}{1 - \beta}. \quad (41)$$

From (37), it is seen that the integrands of (35) and (36) have poles at $\eta = \pm\sqrt{k_{mp}^2 - \zeta^2}$ and $\eta = \pm\sqrt{k_{m0}^2 - \zeta^2}$. The contour of integration is along the real axis of the complex η plane. If the poles of the integrand are on the real axis, it can easily be shown that the radiation condition requires the contour to be indented above the poles on the negative real axis and below those on the positive real axis. The integration with respect to η in (35) and (36) is easily accomplished by closing the contour in the upper half-plane for $z>0$ and in the lower half-plane for $z<0$. The result of such an integration is the following:

$$H_y(x, z) = H_{ym0}(x, z) + H_{ymp}(x, z) \quad (42)$$

where

$$H_{ym0}(x, z) = -\frac{\omega\epsilon_0\epsilon}{4\pi\epsilon_1} \frac{J_0}{(1 - \beta)} \frac{k_{m0}^2 - k_a^2}{k_{m0}^2 - k_{mp}^2}$$

$$\cdot \int_{-\infty}^{\infty} \frac{\exp\left\{i\zeta x + i\sqrt{k_{m0}^2 - \zeta^2}\,|z|\right\}}{\sqrt{k_{m0}^2 - \zeta^2}} d\zeta \quad (43)$$

$$H_{ymp}(x, z) = -\frac{\omega\epsilon_0\epsilon}{4\pi\epsilon_1} \frac{J_0}{(1 - \beta)} \frac{k_{mp}^2 - k_a^2}{k_{mp}^2 - k_a^2}$$

$$\cdot \int_{-\infty}^{\infty} \frac{\exp\left\{i\zeta x + i\sqrt{k_{mp}^2 - \zeta^2}\,|z|\right\}}{\sqrt{k_{mp}^2 - \zeta^2}} d\zeta \quad (44)$$

and

$$p(x, z) = p_{m0}(x, z) + p_{mp}(x, z) \quad (45)$$

where

$$p_{m0}(x, z) = Z_{m0} H_{ym0}(x, z) \quad (46a)$$

$$p_{mp}(x, z) = Z_{mp} H_{ymp}(x, z) \quad (46b)$$

and

$$Z_{m0} = \frac{\omega_p^2 B_0}{\omega^2 - \omega_p^2} \frac{k_{m0}^2}{k_{m0}^2 - k_a^2} \quad (47a)$$

$$Z_{mp} = \frac{\omega_p^2 B_0}{\omega^2 - \omega_p^2} \frac{k_{mp}^2}{k_{mp}^2 - k_a^2}. \quad (47b)$$

The subscripts m0 and mp in (38), (39), and (42)–(47) are used to indicate modified optical and modified plasma modes, respectively. It has been stated before that the small amplitude waves which separate out naturally as plasma and optical modes, interact with each other on the application of an external magnetic field. However, they can be identified as modified plasma and optical modes from an examination of their behavior in the limiting cases of vanishing external magnetic field and infinite source frequency. When there is no applied magnetic field, (34), together with (9) and (14), gives $\beta = 0$. Hence, from (38)–(41), it is clear that

$$k_{m0} = k_{e0}; \qquad k_{mp} = k_{a0} \quad (48)$$

where k_{e0} and k_{a0} are, respectively, the values of k_e and k_a in the absence of the external magnetic field. It follows from (29) and (30), after use of (8) and (12)–(14) that

$$k_{e0}^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \qquad k_{a0}^2 = \frac{\omega^2}{a^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right). \quad (49)$$

After comparison with (21a) and (21b), it becomes obvious that k_{m0} and k_{mp} reduce, respectively, to the wave numbers of optical and plasma modes in the limiting case of zero magnetic field. Hence, k_{m0} and k_{mp} may be identified for convenience as the wave numbers corresponding to modified optical and plasma modes, respectively. It is to be noted, however, that in an unbounded plasma the plasma mode is not excited in the absence of an external magnetic field. In that case, since $B_0 = 0$ and $k_{mp} = k_a = k_{a0}$, it follows from (44), (46), and (47) that $H_{ymp} = p_{m0} = p_{mp} = 0$. The same result also follows from (21a) and (21b) directly. Since there is no source term in the wave equation (21a) for pressure, and since there is no boundary to couple $p(x, z)$ with $H_y(x, z)$, it follows that $p(x, z) = 0$ in the absence of an external magnetic field.

The identification of modified optical and modified plasma modes is also possible from an examination of the behavior of the corresponding wave numbers in the limit of infinite frequency. The use of (8), and (12)–(14) together with (29) and (30) shows that in the limit as ω

tends to infinity, $k_e=\omega/c$ and $k_a=\omega/a$. Also, since $\beta=0$ in this limit, it follows from (38)–(41) that $k_{m0}=\omega/c$ and $k_{mp}=\omega/a$. Since in the limit of infinite frequency k_{m0} and k_{mp} become, respectively, equal to the wave numbers of the transverse optical and longitudinal plasma waves in the plasma without electronic effects, k_{m0} and k_{mp} may be called modified optical and plasma modes, respectively. The examination of (44), (46), and (47) shows that $H_{ymp}(x, z)=p_{mp}(x, z)=p_{m0}(x, z)=0$ and hence, the amplitude of the modified plasma mode also vanishes in the limit of infinite frequency.

It has been shown that there are two modes of wave propagation which have been called modified optical and plasma modes. The modified optical mode has associated pressure variations and the modified plasma mode has associated with it a transverse component of the magnetic field. These modes in the limiting cases of zero external magnetic field and infinite source frequency reduce to the usual optical mode and the plasma mode with no associated component of the magnetic field.

For completing the determination of $H_y(x, z)$ and $p(x, z)$, the integrations with respect to ζ remain to be carried out. It is seen that the integral in (43) has branch points at $\zeta=\pm k_{m0}$ and those in (44) at $\zeta=\pm k_{mp}$. The contour of integration in all the cases is along the real axis of the ζ plane. The branch-cuts in the ζ plane have to be chosen to fulfill the radiation condition which requires an outward flow of power from the source at large distances. The contour for the integral in (43) is indented above the singularity $\zeta=-k_{m0}$ and below the singularity $\zeta=k_{m0}$. The branch-cuts at $\zeta=-k_{m0}$ and $\zeta=k_{m0}$ are taken parallel to the imaginary axis, respectively in the lower and the upper half-planes. Similarly, the contour for the integral in (44) is indented above the singularity $\zeta=-k_{mp}$ and below the singularity $\zeta=k_{m0}$. The branch-cuts at $\zeta=\mp k_{mp}$ are also chosen in the same manner as in the previous case. The integrals [(43) and (44)] are easily evaluated and the following result obtained:

$$H_{ym0}(x, z) = -\frac{\omega\epsilon_0\epsilon}{4\epsilon_1}\frac{J_0}{(1-\beta)}\frac{k_{m0}^2-k_a^2}{k_{m0}^2-k_{mp}^2}H_0^{(1)}[k_{m0}\rho] \quad (50)$$

$$H_{ymp}(x, z) = \frac{\omega\epsilon_0\epsilon}{4\epsilon_1}\frac{J_0}{(1-\beta)}\frac{k_{mp}^2-k_a^2}{k_{m0}^2-k_{mp}^2}H_0^{(1)}[k_{mp}\rho] \quad (51)$$

where the polar coordinates defined by

$$x = \rho\cos\theta; \qquad z = \rho\sin\theta, \quad (52)$$

are introduced. The choice of the branch-cuts in the evaluation of the integrals [(43) and (44)] ensures outward traveling phase fronts. But this condition does not necessarily imply outward flow of power. Therefore, it remains to be verified that the present choice of branch-cuts does indeed lead to the fulfillment of the radiation condition.

Dispersion Curves

For the evaluation of power, it is necessary to examine the nature of the dispersion curves $\omega-k_{m0}$ and $\omega-k_{mp}$. It is convenient to normalize ω, ω_c, k_{m0}, and k_{mp} in the following manner:

$$\Omega = \frac{\omega}{\omega_p}; \; R = \frac{\omega_c}{\omega_p}; \; K_{m0} = \frac{\sqrt{ac}}{\omega p}k_{m0}; \; K_{mp} = \frac{\sqrt{ac}}{\omega p}k_{mp}. \quad (53)$$

By using (8), (12)–(14), (29), (30), and (53) in (38)–(41), it can be shown that

$$K_{m0} = [U_1 - \sqrt{U_1^2 - W_1}]^{1/2} \quad (54a)$$

$$K_{mp} = [U_1 + \sqrt{U_1^2 - W_1}]^{1/2} \quad (54b)$$

where

$$U_1 = \frac{ac}{\omega_p^2}U = \frac{c}{2a}[\Omega^2 - (1+R^2)] \quad (55)$$

$$W_1 = \frac{a^2c^2}{\omega_p^4}W = \Omega^4 - \Omega^2(2+R^2) + 1. \quad (56)$$

In obtaining (56), a^2/c^2 has been neglected in comparison with unity. This is legitimate, since the ratio of the acoustic to the electromagnetic wave velocities a/c is of the order of 10^{-4}. From (56), it is seen that W_1 becomes zero at two values of Ω, namely Ω_1 and Ω_3, and these are given by

$$\Omega_1^2 = 1 + \frac{R^2}{2} - \sqrt{\frac{R^4}{4} + R^2} \quad (57)$$

$$\Omega_3^2 = 1 + \frac{R^2}{2} + \sqrt{\frac{R^4}{4} + R^2}. \quad (58)$$

Also U_1 becomes zero at $\Omega=\Omega_2$ given by

$$\Omega_2^2 = 1 + R^2. \quad (59)$$

From (57), (58), and (59), it is obvious that $\Omega_1<\Omega_2<\Omega_3$. It is clear that $U_1 \lesseqgtr 0$ according to whether $\Omega \lesseqgtr \Omega_2$ and that $W_1<0$ for $\Omega_1<\Omega<\Omega_2$ and $W_1>0$ for $0<\Omega<\Omega_1$ and $\Omega_3<\Omega<\infty$. In the frequency range $0<\Omega<\Omega_1$, $W_1>0$, and therefore $|\sqrt{U_1^2-W_1}|<|U_1|$. Also $U_1^2>W_1$ as a result of the large value of the factor c/a in (55). Hence, $U_1\mp\sqrt{U_1^2-W_1}$ are both real and have the same sign as U_1 which is negative. Therefore, K_{m0} and K_{mp} are purely imaginary and the two modes are nonpropagating. In the frequency range $\Omega_1<\Omega<\Omega_3$, $W_1<0$, and $|\sqrt{U_1^2-W_1}|>|U_1|$, and therefore, K_{mp} is positive real and K_{m0} is purely imaginary and this shows that only the plasma mode propagates in this range of Ω. In the frequency range $\Omega_3<\Omega<\infty$, $U_1^2>W_1$. Also $W_1>0$ and $|\sqrt{U_1^2-W_1}|<|U_1|$. Therefore, both $U_1\mp\sqrt{U_1^2-W_1}$ are real and of the same sign as U_1. Because $U_1>0$ in this frequency range, K_{m0} and K_{mp}

are both positive real resulting in a pass band for both the plasma and optical modes.

In Figs. 1, 2, and 3 the dispersion curves $\Omega - K_{m0}$ and $\Omega - K_{mp}$ are plotted for three values of R, namely, 0.5, 1, and 5. The ratio a/c of the electron sound to the electromagnetic wave velocity is taken to be 10^{-4}. When $R=0$, that is, in the absence of an external magnetic field, both modes are cut off below $\Omega=1$. With the application of an external magnetic field, the cut-off frequency Ω_3 of the optical mode increases and that of the plasma mode Ω_1 decreases. The phase velocity of the modified plasma mode remains approximately at the value corresponding to that of the electromagnetic wave and rapidly decreases, in the neighborhood of $\Omega=\Omega_2$, to that of the acoustic wave in the electron gas. The cut-off frequency Ω_3 of the modified optical mode continuously increases and that of the modified plasma mode Ω_1, continuously decreases when the external magnetic field is increased. The frequency band $(\Omega_3-\Omega_1)$ in which only the modified plasma mode can propagate is approximately equal to R and hence, increases with an increase in the external magnetic field.

Radiated Power in the Optical and the Plasma Modes

It is desired to calculate the time-averaged power radiated by the line source in the optical and the plasma modes. For this purpose a generalized Poynting vector for a compressible plasma must be derived. It is convenient to start with the following time-dependent forms of (2)–(5):

$$mN_0 \frac{\partial}{\partial t} \boldsymbol{v} = N_0 e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) - \nabla p \qquad (60)$$

$$a^2 m N_0 \nabla \cdot \boldsymbol{v} = -\frac{\partial p}{\partial t} \qquad (61)$$

$$\nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} - \boldsymbol{J}_m \qquad (62)$$

$$\nabla \times \boldsymbol{H} = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + N_0 e \boldsymbol{v}. \qquad (63)$$

Let

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} + p\boldsymbol{v}. \qquad (64)$$

Then

$$\nabla \cdot \boldsymbol{S} = \boldsymbol{H} \cdot \nabla \times \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{H} + \nabla p \cdot \boldsymbol{v} + p \nabla \cdot \boldsymbol{v}. \qquad (65)$$

On substituting for $\nabla \times \boldsymbol{E}$, $\nabla \times \boldsymbol{H}$, ∇p and $\nabla \cdot \boldsymbol{v}$ respectively from (62), (63), (60), and (61) in (65) and noting that $\boldsymbol{v} \times \boldsymbol{B} \cdot \boldsymbol{v} = 0$, it follows that

$$\nabla \cdot \boldsymbol{S} = -\frac{\partial}{\partial t}\left[\frac{\mu_0}{2} H^2 + \frac{\epsilon_0}{2} E^2 + \frac{mN_0}{2} v^2 + \frac{1}{a^2 m N_0} p^2\right] - \boldsymbol{H} \cdot \boldsymbol{J}_m. \qquad (66)$$

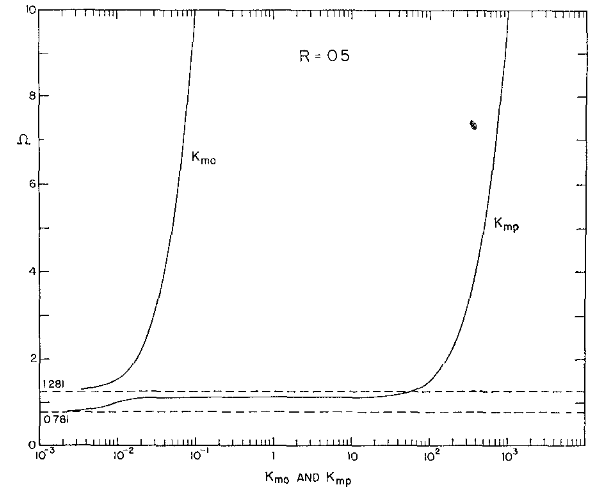

Fig. 1—Dispersion curve for $R=0.5$.

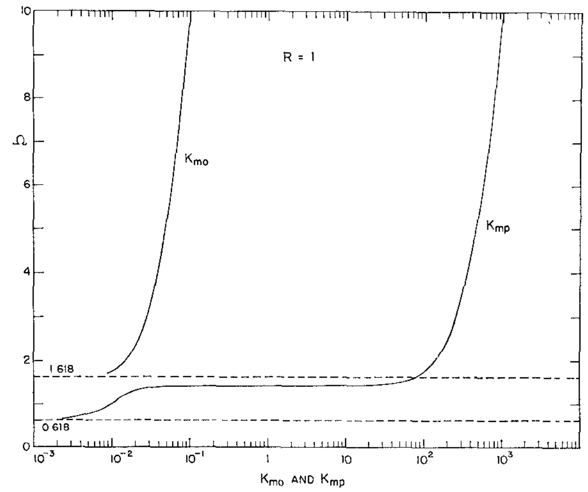

Fig. 2—Dispersion curve for $R=1$.

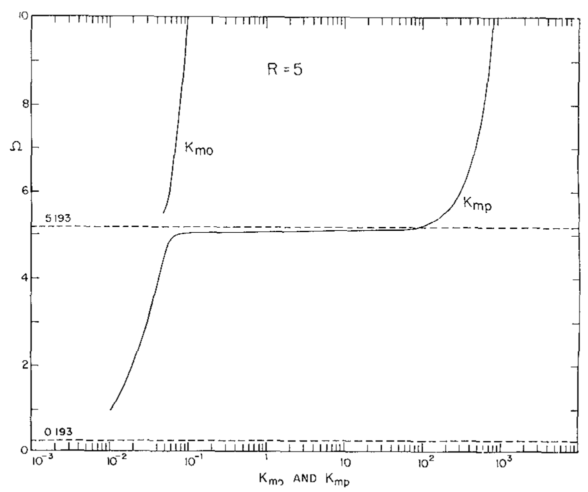

Fig. 3—Dispersion curve for $R=5$.

The integration of (66) throughout the volume V enclosed by the surface A yields

$$\int_A \boldsymbol{S}\cdot\hat{n}dA + \frac{\partial}{\partial t}\int\left[\frac{\mu_0}{2}|H|^2 + \frac{\epsilon_0}{2}|E|^2 + \frac{mN_0}{2}|v|^2 + \frac{1}{a^2mN_0}|p|^2\right]dV = -\int_V \boldsymbol{H}\cdot\boldsymbol{J}_m dV. \quad (67)$$

The term inside the square brackets in (67) gives the sum of the densities of magnetic, electric, kinetic, and potential energies, and hence, the second term on the right side of (67) represents the rate of increase of the total energy inside the volume V. The term on the right-hand side of (67) is the rate of supply of energy by the source. The requirement of energy balance immediately shows that the first term on the left-hand side is the rate of outward flow of energy through the area A. Hence the vector $\boldsymbol{S}$ represents the outward power flow per unit area. In the case of harmonic time dependence, the outward time-averaged power flow through unit area is easily seen from (64) to be given by

$$\boldsymbol{S} = \text{Re}\,\tfrac{1}{2}[\boldsymbol{E}\times\boldsymbol{H}^* + pv^*]. \quad (68)$$

In order to be able to speak of the power radiated separately in the optical and the plasma modes, it is necessary to show that the two modes are orthogonal. This is, that

$$\int_A [\boldsymbol{E}_{mp}\times\boldsymbol{H}_{m0}^* + p_{mp}\overset{*}{\boldsymbol{v}}_{m0}]\cdot d\boldsymbol{A} = 0 \quad (69)$$

where the integral is taken over a surface enclosing the source. In order to establish (69), the following Fourier-transformed expressions for $\boldsymbol{E}_{mp}$, $\boldsymbol{H}_{m0}$, p_{mp}, and $\boldsymbol{v}_{m0}$ are substituted in the integral on the right-hand side of (69):

$$\boldsymbol{E}_{mp} = \frac{1}{2\pi}\int \boldsymbol{E}_{mp}(\zeta, k_{mp})\cdot\exp\left\{i\zeta x + i\sqrt{k_{mp}^2-\zeta^2}\,|z|\right\}d\zeta \quad (70)$$

$$\boldsymbol{H}_{m0} = \frac{1}{2\pi}\int \boldsymbol{H}(\zeta, k_{m0})\cdot\exp\left\{i\zeta x + i\sqrt{k_{m0}^2-\zeta^2}\,|z|\right\}d\zeta \quad (71)$$

$$p_{mp} = \frac{1}{2\pi}\int p_{mp}(\zeta, k_{mp})\cdot\exp\left\{i\zeta x + i\sqrt{k_{mp}^2-\zeta^2}\,|z|\right\}d\zeta \quad (72)$$

$$v_{m0} = \frac{1}{2\pi}\int v_{m0}(\zeta, k_{m0})\cdot\exp\left\{i\zeta x + i\sqrt{k_{m0}^2-\zeta^2}\,|z|\right\}d\zeta. \quad (73)$$

The result is

$$\begin{aligned}\int_A [\boldsymbol{E}_{mp}\times\boldsymbol{H}_{m0}^* + p_{mp}\boldsymbol{v}_{m0}^*]\cdot d\boldsymbol{A} &= \frac{1}{(2\pi)^2}\int_A\int_{\zeta_1}\int_{\zeta_2}[\boldsymbol{E}_{mp}(\zeta_1, k_{mp})\times\boldsymbol{H}_{m0}^*(\zeta_2, k_{m0}) + p_{mp}(\zeta_1, k_{mp})\boldsymbol{v}_{m0}^*(\zeta_2, k_{m0})]\\ &\quad\cdot\exp\left\{i(\zeta_1-\zeta_2)x + i(\sqrt{k_{mp}^2-\zeta_1^2}-\sqrt{k_{m0}^2-\zeta_2^2})\,|z|\right\}\cdot d\boldsymbol{A}d\zeta_1 d\zeta_2\\ &= \int_{\zeta_1}\int_{\zeta_2}[\boldsymbol{E}_{mp}(\zeta_1, k_{mp})\times\boldsymbol{H}_{m0}^*(\zeta_2, k_{m0}) + p_{mp}(\zeta_1, k_{mp})\boldsymbol{v}_{m0}^*(\zeta_2, k_{m0})]\delta(\zeta_1-\zeta_2)\delta[\sqrt{k_{mp}^2-\zeta_1^2}-\sqrt{k_{m0}^2-\zeta_2^2}]d\zeta_1 d\zeta_2\\ &= \int_\zeta [\boldsymbol{E}_{mp}(\zeta, k_{mp})\times\boldsymbol{H}_{m0}^*(\zeta, k_{m0}) + p_{mp}(\zeta, k_{mp})\boldsymbol{v}_{m0}^*(\zeta, k_{m0})]\delta[\sqrt{k_{mp}^2-\zeta^2}-\sqrt{k_{m0}^2-\zeta^2}]d\zeta. \end{aligned} \quad (74)$$

Since the wave numbers k_{m0} and k_{mp} are always different, it follows that (74) is equal to zero and hence, the orthogonality relation (69) is established.

If use is made of (52), (10), (11), (16), and (17), and it is noted that the field components are independent of the angular variable, it is easily shown that

$$E_\rho = -\frac{\epsilon_2}{\omega\epsilon_0\epsilon}\frac{\partial H_y}{\partial\rho} - \frac{(\epsilon_1-\epsilon)}{N_0 e\epsilon}\frac{\partial p}{\partial\rho} \quad (75)$$

$$E_\theta = \frac{i\epsilon_1}{\omega\epsilon_0\epsilon}\frac{\partial H_y}{\partial\rho} + \frac{i\epsilon_2}{N_0 e\epsilon}\frac{\partial p}{\partial\rho} \quad (76)$$

$$v_\rho = -\frac{ie\omega_c}{\omega^3 m\epsilon_0\epsilon\alpha}\frac{\partial H_y}{\partial\rho} - \frac{i\left[1-\frac{\omega_p^2}{\omega^2}\right]}{\omega\epsilon\alpha m N_0}\frac{\partial p}{\partial\rho} \quad (77)$$

$$v_\theta = -\frac{e\left[1-\frac{\omega_p^2}{\omega^2}\right]}{\omega^2 m\epsilon_0\epsilon\alpha}\frac{\partial H_y}{\partial\rho} - \frac{\omega_c}{\omega^2\epsilon\alpha m N_0}\frac{\partial p}{\partial\rho}. \quad (78)$$

The total powers radiated by the line source in the optical and plasma modes are given by

$$P_{m0} \quad \int_0^{2\pi} S_{m0}\cdot\hat{\rho}\rho d\theta = \text{Re}\,\pi\rho[-E_{\theta m0}H_{ym0}^* + p_{m0}v_{\rho m0}^*] \quad (79)$$

$$P_{mp} = \int_0^{2\pi} S_{mp}\cdot\hat{\rho}\rho d\theta = \text{Re}\,\pi\rho[-E_{\theta mp}H_{\ mp}^* + p_{mp}v_{\rho mp}^*]. \quad (80)$$

The substitution for H_y and p, respectively, from (50), (51), and (46a) and (46b) in (76) and (77) gives the expressions for E_θ and v_ρ. The insertion of the expressions for E_θ, H_y, p, and v_ρ in (79) and (80), respectively, for the optical and plasma modes and the replacement of the Hankel function by the first term in its asymptotic expansion, leads to the following result for the power radiated in the two modes:

$$\tilde{P}_{m0} = \frac{P_{m0}}{\frac{\omega}{2}\epsilon_0 J_0^2} = \left[\frac{\epsilon_1}{\epsilon} - \frac{\epsilon_2}{\epsilon}\frac{\omega\omega_c}{\omega^2-\omega_p^2}\frac{k_{m0}^2}{k_{m0}^2-k_a^2} + \frac{\epsilon_2\omega\omega_c k_{m0}^2 k_a^2}{\epsilon(\omega^2-\omega_p^2)(k_{m0}^2-k_a^2)^2}\right] \times \left[\frac{\epsilon}{2\epsilon_1}\frac{1}{1-\beta}\frac{k_{m0}^2-k_a^2}{k_{m0}^2-k_p^2}\right]^2 \quad \text{for } \Omega > \Omega_3 \tag{81}$$

$$\tilde{P}_{mp} = \frac{P_{mp}}{\frac{\omega}{2}\epsilon_0 J_0^2} = \left[\frac{\epsilon_1}{\epsilon} - \frac{\epsilon_2\omega\omega_c}{\epsilon(\omega^2-\omega_p^2)}\frac{k_{mp}^2}{k_{mp}^2-k_a^2} + \frac{\epsilon_2}{\epsilon}\frac{\omega\omega_c k_{mp}^2 k_a^2}{(\omega^2-\omega_p^2)(k_{mp}^2-k_a^2)^2}\right] \times \left[\frac{\epsilon}{2\epsilon_1}\frac{1}{1-\beta}\frac{k_{mp}^2-k_a^2}{k_{m0}^2-k_p^2}\right]^2 \quad \text{for } \Omega > \Omega_1. \tag{82}$$

Care must be exercised when the radiated power is evaluated for the values $\Omega=\Omega_1$ and $\Omega=\Omega_3$. The expressions (81) and (82) are not used for this purpose. It is shown below that no power is radiated for these two values Ω. It is easily shown with the help of (34), (13), (14), (8), and (53) that

$$1-\beta = \frac{(\Omega^2-\Omega_1^2)(\Omega^2-\Omega_3^2)}{(\Omega^2-1)(\Omega^2-R^2-1)}, \tag{83}$$

where the expressions for Ω_1^2 and Ω_3^2 are given respectively in (57) and (58). For $\Omega=\Omega_1$ and $\Omega=\Omega_3$, $(1-\beta)$ is seen to be zero from (83), and as a consequence, the expression (37) for Δ is no longer valid. With the help of (29), (30), (12)–(14), (8), and (53), it is seen that

$$k_a^2 = \frac{\omega_p^2}{a^2}\frac{[\Omega^2-\Omega_1^2][\Omega^2-\Omega_3^2]}{[\Omega^2-1]} \tag{84}$$

$$k_e^2 = \frac{\omega_p^2}{c^2}\frac{[\Omega^2-\Omega_1^2][\Omega^2-\Omega_3^2]}{[\Omega^2-R^2-1]} \tag{85}$$

and

$$\epsilon = \frac{[\Omega^2-\Omega_1^2][\Omega^2-\Omega_3^2]}{\Omega^2[\Omega^2-R^2]}. \tag{86}$$

For the values of $\Omega=\Omega_1$ and $\Omega=\Omega_3$, from (84), (85), and (86) it is seen that $k_a^2=k_e^2=\epsilon=0$. These values when substituted in (27) and (28), immediately give

$$H_y(x,z) = p(x,z) = 0 \qquad \text{for } \Omega=\Omega_1 \text{ and } \Omega_3. \tag{87}$$

Hence, no power is radiated for the values $\Omega=\Omega_1$ and $\Omega=\Omega_3$.

With the help of (8), (12)–(14), (29), (30), (34), (38)–(41), (53), and (57)–(59) the expression for $\tilde{P}_{m0}$ given in (81) may be simplified to yield the following result:

$$\tilde{P}_{m0} = \left[\frac{\Omega^2(\Omega^2-\Omega_2^2)}{(\Omega^2-\Omega_1^2)(\Omega^2-\Omega_3^2)} + \frac{R^2\Omega^2}{(\Omega^2-\Omega_1^2)(\Omega^2-\Omega_3^2)(\Omega^2-1)}\cdot\left\{-\frac{N}{N-1}+\frac{N}{(N-1)^2}\right\}\right] \times \left[\frac{(\Omega^2-1)}{2\Omega^2}\frac{N-1}{N-M}\right]^2 \quad \text{for } \Omega > \Omega_3, \tag{88}$$

where

$$M = u + \sqrt{u^2-w} \tag{89}$$

$$N = u - \sqrt{u^2-w} \tag{90}$$

$$u = \frac{(\Omega^2-1)(\Omega^2-\Omega_2^2)}{2(\Omega^2-\Omega_1^2)(\Omega^2-\Omega_3^2)}\left[1+10^{-8}\frac{(\Omega^2-1)}{(\Omega^2-\Omega_2^2)}\right] \tag{91}$$

and

$$w = \frac{10^{-8}(\Omega^2-1)^2}{(\Omega^2-\Omega_1^2)(\Omega^2-\Omega_3^2)}. \tag{92}$$

The velocity ratio a/c is taken to be equal to 10^{-4}. In the range $\Omega_3<\Omega<\infty$, where the optical mode propagates, the second term inside the square brackets of (91) can be omitted and u is seen to be very large compared to w. Hence, (89) and (90) may be approximated as follows:

$$M \doteq u + |u| - \frac{w}{2|u|} \doteq 2u \doteq \frac{(\Omega^2-1)(\Omega^2-\Omega_2^2)}{(\Omega^2-\Omega_1^2)(\Omega^2-\Omega_3^2)} \tag{93}$$

$$N \doteq u - |u| + \frac{w}{2|u|} \doteq \frac{w}{2|u|} \doteq \frac{10^{-8}(\Omega^2-1)}{2(\Omega^2-\Omega_2^2)}. \tag{94}$$

The results in (93) and (94) follow from the fact that u is positive in the range $\Omega_3<\Omega<\infty$, as can be seen from (91). From (93) and (94), N is seen to be very much smaller in comparison to M and hence, can be neglected in (88). Therefore,

$$\tilde{P}_{m0} = \frac{\Omega^2(\Omega^2-\Omega_2^2)}{(\Omega^2-\Omega_1^2)(\Omega^2-\Omega_3^2)}\left[\frac{(\Omega^2-1)}{2\Omega^2}\frac{1}{M}\right]^2 = \frac{(\Omega^2-\Omega_1^2)(\Omega^2-\Omega_3^2)}{4\Omega^2(\Omega^2-\Omega_2^2)}. \tag{95}$$

It is evident from (95) that $\tilde{P}_{m0}$ is positive in the range $\Omega_3<\Omega<\infty$ and this ensures a net outward flow of power from the source. Thus, the radiation condition is satisfied, and the choice of the branch-cuts used in the evaluation of the integral in (43) is justified. The power $\tilde{P}_{m0}$ radiated in the modified optical mode may be calculated from the expression (95) for the range $\Omega_3<\Omega<10$ and for different values of the parameter R. The results are given in a graphical form in Fig. 4. It is seen from the figure that power radiated in the modified optical mode at given frequency becomes smaller as the applied magnetic field is increased. Also, at a given external magnetic field, the power in this mode rapidly increases from zero as the frequency is increased from the cut-off frequency Ω_3 and reaches an asymptotic value which is the same as when there is no external magnetic field.

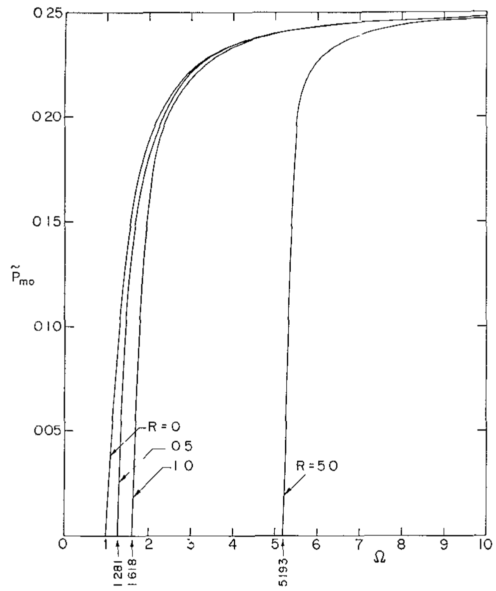

Fig. 4—Power in the modified optical mode for different values of R.

In the same manner as (81), (82) may be simplified to yield

$$\tilde{P}_{mp} = \left[\frac{\Omega^2(\Omega^2-\Omega_2{}^2)}{(\Omega^2-\Omega_1{}^2)(\Omega^2-\Omega_3{}^2)} + \frac{R^2\Omega^2}{(\Omega^2-\Omega_1{}^2)(\Omega^2-\Omega_3{}^2)(\Omega^2-1)}\left\{-\frac{M}{M-1}+\frac{M}{(M-1)^2}\right\}\right] \times \left[\frac{(\Omega^2-1)}{2\Omega^2}\,\frac{M-1}{N-M}\right]^2 \quad \text{for } \Omega>\Omega_1. \tag{96}$$

Two cases may now be distinguished according to whether u is positive or negative. For Ω not in the neighborhood Ω_2 and u positive, (93) and (94) are valid and, therefore, N can be neglected in comparison with M. Hence, it follows from (90), (93) and (96) that

$$\tilde{P}_{mp} = \frac{R^2}{4\Omega^2(\Omega^2-\Omega_2{}^2)} \quad \text{for } 1<\Omega<\Omega_2 \text{ and } \Omega_3<\Omega<\infty. \tag{97}$$

It is seen from (91) that u is positive for the ranges $1<\Omega<\Omega_2$ and $\Omega_3<\Omega<\infty$. From (97), it is obvious that $\tilde{P}_{mp}$ is negative in the range $1<\Omega<\Omega_2$ and positive in the range $\Omega_3<\Omega<\infty$. When Ω is not in the neighborhood of Ω_2 and when u is negative, it is easily derived from (93) and (94) that

$$M = -\frac{w}{2|u|} = \frac{10^{-8}(\Omega^2-1)}{2(\Omega^2-\Omega_2{}^2)} \tag{98}$$

$$N \doteq -2|u| = \frac{(\Omega^2-1)(\Omega^2-\Omega_2{}^2)}{(\Omega^2-\Omega_1{}^2)(\Omega^2-\Omega_3{}^2)}\cdot \tag{99}$$

From (98) and (99), it is seen that M is very small compared to N and 1 and, therefore, it can be neglected in (96) with the following result:

$$\tilde{P}_{mp} \doteq \frac{\Omega^2(\Omega^2-\Omega_2{}^2)}{(\Omega^2-\Omega_1{}^2)(\Omega^2-\Omega_3{}^2)}\left[\frac{(\Omega^2-1)}{2\Omega^2}\,\frac{1}{N}\right]^2. \tag{100}$$

The substitution of the expression for N from (99) in (100) yields

$$\tilde{P}_{mp} \simeq \frac{(\Omega^2-\Omega_1{}^2)(\Omega^2-\Omega_3{}^2)}{4\Omega^2(\Omega^2-\Omega_2{}^2)} \quad \text{for } \Omega_1<\Omega<1 \text{ and } \Omega_2<\Omega<\Omega_3. \tag{101}$$

From (91), it is obvious that u is negative in the ranges $\Omega_1<\Omega<1$ and $\Omega_2<\Omega<\Omega_3$. It follows from (101) that $\tilde{P}_{mp}$ is positive for the range $\Omega_1<\Omega<1$ and negative for $\Omega_2<\Omega<\Omega_3$.

When Ω is in the near neighborhood of Ω_2, the following valid approximation for u can be made from (91):

$$u = \frac{10^{-8}(\Omega^2-1)^2}{2(\Omega^2-\Omega_1{}^2)(\Omega^2-\Omega_3{}^2)} = \frac{w}{2}\cdot \tag{102}$$

Since $u \ll 1$, it follows from (89) and (90) that

$$M = -N \doteq \sqrt{-w} = \frac{10^{-4}(\Omega^2-1)}{[(\Omega^2-\Omega_1{}^2)(\Omega_3{}^2-\Omega^2)]^{1/2}}\cdot \tag{103}$$

It is seen that M, $N \ll 1$, and hence, for Ω in the near neighborhood of Ω_2, (96) may be simplified to yield

$$\tilde{P}_{mp} = \frac{-R^2 10^4}{8\Omega^2[(\Omega^2-\Omega_1{}^2)(\Omega_3{}^2-\Omega^2)]^{1/2}}\cdot \tag{104}$$

From (104), $\tilde{P}_{mp}$ is seen to be negative even when Ω is near Ω_2. Thus, from (97), (101) and (104), $\tilde{P}_{mp}$ is positive for $\Omega_1<\Omega<1$ and $\Omega_3<\Omega<\infty$ and negative for $1<\Omega<\Omega_3$.

Since $\tilde{P}_{mp}$ is positive for the ranges $\Omega_1<\Omega<1$ and $\Omega_3<\Omega<\infty$, the radiation condition is fulfilled and hence

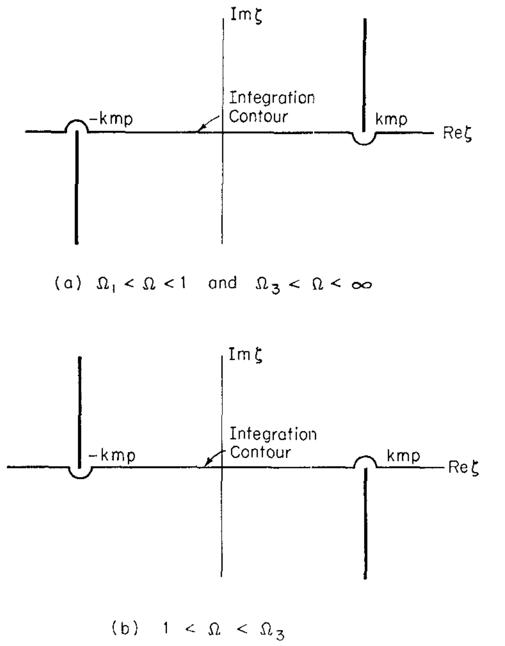

Fig. 5—Branch-cuts in the ζ plane for the modified plasma mode.

the choice of branch-cuts [see Fig. 5(a)] used in the evaluation of the integral (44) is correct. But for $1<\Omega<\Omega_3$, $\tilde{P}_{mp}$ is negative and, therefore, the radiation condition is not satisfied. In order to satisfy the radiation condition even for $1<\Omega<\Omega_3$, it is necessary to use the branch-cuts as illustrated in Fig. 5(b). The new choice of branch-cuts will lead to $H_0^{(2)}$ in (50) and (51) instead of to $H_0^{(1)}$. A negative sign is introduced in (82) as a result and $\tilde{P}_{mp}$ turns out to be positive, ensuring the fulfillment of the radiation condition.

The power radiated in the modified plasma mode may be evaluated for values of Ω ranging from Ω_1 to 10, with the help of (97), (101), and (104) and the results are plotted in Fig. 6. From the figure it is seen that as Ω increases from Ω_3 the power in the plasma mode rapidly decreases. On the other hand, the power in the optical mode rapidly increases and reaches an asymptotic value. For certain ranges of frequencies, the power in the modified plasma mode is higher than that radiated by the line source in free space. Also, there is a peak at $\Omega=\Omega_2$ and from (104) it can easily be shown that this peak has its maximum value when $\omega/\omega_p=\omega/\omega_c=\sqrt{2}$. With the introduction of dissipative effects in the plasma, this peak is reduced in magnitude, but nevertheless, the possibility of obtaining more power from the line source than in free space exists, and this appears to be significant.

In this investigation the motion of the ions has been neglected in comparison with that of the electrons and as a consequence, the interaction between the sound waves and the plasma oscillations cannot be determined. This aspect of the problem is under investigation and will be the subject of a subsequent paper.

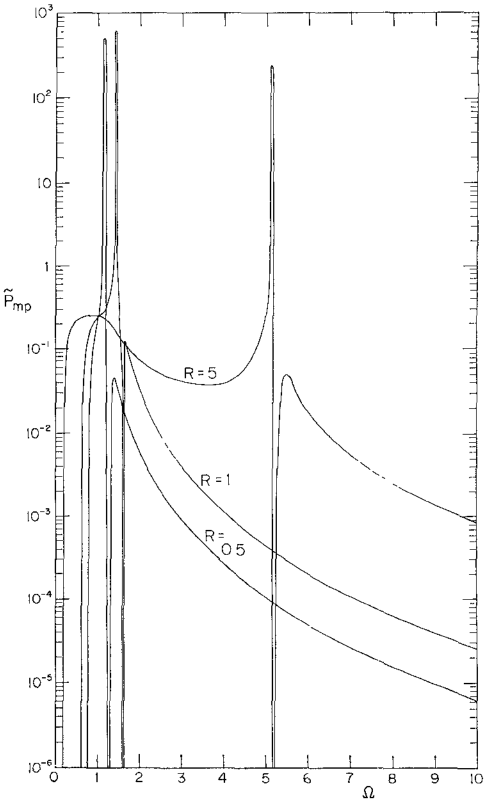

Fig. 6—Power radiated in the modified plasma mode for different values of R.

Appendix

The following elegant method of solving the coupled equations (18) and (19) has been pointed out to the author by Prof. J. Shmoys of Polytechnic Institute of Brooklyn, N. Y.

After the substitution of $\nabla^2 p$ from (18) into (19) and $\nabla^2 H_y$ from (19) into (18), it follows that

$$(1-\beta)\nabla^2 H_y - \frac{\omega\epsilon_0\epsilon_2}{N_0 e\epsilon_1}\,\frac{\omega^2\alpha\epsilon}{a^2\left(1-\dfrac{\omega_p^2}{\omega^2}\right)}\,p + k_e^2 H_y = \frac{-i\omega\epsilon_0\epsilon}{\epsilon_1} J_0\delta(x)\delta(z) \quad (105)$$

$$(1-\beta)\nabla^2 p + k_a^2 p + \frac{k_e^2\omega_p^2 B_0}{\omega^2-\omega_p^2} H_y = -\frac{i\omega\epsilon_0\epsilon}{\epsilon_1}\,\frac{\omega_p^2 B_0}{\omega^2-\omega_p^2} J_0\delta(x)\delta(z). \quad (106)$$

The two equations (105) and (106) can be written down as a single equation for the vector

$$\begin{pmatrix} H_y \\ p \end{pmatrix}$$

in the following manner:

$$\nabla^2\begin{bmatrix} H_y \\ p \end{bmatrix} + [M]\begin{bmatrix} H_y \\ p \end{bmatrix} = [S] \tag{107}$$

where

$$[M] = \frac{1}{1-\beta}\begin{bmatrix} k_e^2 & \dfrac{-\omega\epsilon_0\epsilon_2}{N_0 e\epsilon_1}\dfrac{\omega^2\alpha\epsilon}{a^2\left(1-\dfrac{\omega_p^2}{\omega^2}\right)} \\ \dfrac{k_e^2\omega_p^2 B_0}{\omega^2-\omega_p^2} & k_a^2 \end{bmatrix} \tag{108}$$

and

$$[S] = -\frac{i\omega\epsilon_0\epsilon}{(1-\beta)\epsilon_1}J_0\delta(x)\delta(z)\begin{bmatrix} 1 \\ \dfrac{\omega_p^2 B_0}{\omega^2-\omega_p^2} \end{bmatrix}. \tag{109}$$

Introduce the following transformation:

$$\begin{bmatrix} H_y \\ p \end{bmatrix} = [T]\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \tag{110}$$

The substitution of (110) in (107) and the premultiplication by the inverse matrix $[T]^{-1}$ leads to

$$\nabla^2\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + [T]^{-1}[M][T]\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = [T]^{-1}[S]. \tag{111}$$

If the matrix $[T]$ is chosen in such a way as to diagonalize $[M]$, the following two uncoupled wave equations are obtained:

$$\nabla^2\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} k_{m0}^2 & 0 \\ 0 & k_{mp}^2 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = [T]^{-1}[S] \tag{112}$$

where k_{m0}^2 and k_{mp}^2, the eigenvalues of $[M]$, are given by the roots of the equation

$$(1-\beta)\lambda^2 - (k_a^2 + k_e^2)\lambda + k_a^2 k_e^2 = 0. \tag{113}$$

From (113) it is clear that k_{m0}^2 and k_{mp}^2 are respectively the same as given in (38) and (39). The evaluation of the inverse matrix $[T]^{-1}$ yields the source term on the left-hand side of (112). Since the source terms are delta functions, the solutions are obviously Hankel functions.

Acknowledgment

The author wishes to thank Profs. R. W. P. King and T. T. Wu for help and encouragement with this research.

Radial-Line Coaxial Filters in the Microwave Region*

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Summary—**Design techniques and a simple empirical formula for the design of band rejection radial-line coaxial filters are presented. The appropriateness of these filters for parametric work is discussed and a particular structure employing these filters to provide a high performance harmonic filter structure for rectangular waveguide is presented.**

I. Introduction

SEVERAL requests for "further information" on radial-line coaxial filters followed the presentation of a paper[1] at the 1961 International Solid State Circuits Conference. This paper is a response to those requests and is intended to provide a practical design technique for the realization of these filters.

The design of coaxial filters in the microwave region above a few gigacycles has not received much attention in the past due to the popularity of rectangular waveguide for use at these frequencies. Coaxial filters in this frequency range have become increasingly important of late, however, due in large part to the advent of multiple frequency circuits employing coaxial lines (often in conjunction with other types of waveguides) which have come about through the application of solid-state art to microwave problems. Parametric amplifiers and frequency multipliers (or dividers) in particular have stringent filtering requirements for which coaxial filters of the type to be discussed in this paper seem particularly appropriate.

In addition, harmonic band rejection filters in rectangular waveguide structures are difficult to design for very good fundamental frequency performance and are often rather poor in their filtering response for one or more of the several harmonic waveguide modes that may be present. The problems associated with these filters can be avoided by accomplishing the filtering in a coaxial line and employing two rectangular wave-guide-to-coaxial line transducers.

* Received July 16, 1962; revised manuscript received September 26, 1962.

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[1] B. C. De Loach, Jr., "Waveguide parametric amplifiers," Digest of Technical Papers, 1961 Internat'l. Solid-State Circuits Conf., Lewis Winner, New York, N. Y.